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On the Convergence of a Modified Kähler-Ricci Flow

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Abstract

We study the convergence of a modified Kähler-Ricci flow defined by Zhou Zhang. We show that the flow converges to a singular metric when the limit class is degenerate. This proves a conjecture of Zhang.

1 Introduction

The Ricci flow was introduced by Richard Hamilton [H] on Riemannian manifolds to study the deformation of metrics. Its analogue in Kähler geometry, the Käher-Ricci flow, has been intensively studied in the recent years. It turns out to be a powerful method to study the canonical metrics on Kähler manifolds. (See, for instance, the papers [C] [CT] [Pe] [PS] [PSSW] [ST1] [ST2] [TZhu] and the references therein.) In a recent paper [Z3], a modified Kähler-Ricci flow was defined by Zhang by allowing the cohomology class to vary artificially. We briefly describe it as follows:

Let X be a closed Kähler manifold of complex dimension n with a Kähler metric ω_0, and let ω_∞ be a real, smooth, closed $(1,1)$-form with $[\omega_\infty]^n = 1$. Let Ω be a smooth volume form on X such that $\int_X \Omega = 1$. Set $\chi = \omega_0 - \omega_\infty$, $\omega_t = \omega_\infty + e^{-t}\chi$. Let $\varphi : [0,\infty) \times X \to \mathbb{R}$ be a smooth function such that $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi > 0$. Consider the following Monge-Ampère flow:

$$\frac{\partial}{\partial t}\varphi = \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}; \qquad \varphi(0,\cdot) = 0. \tag{1.1}$$

Then the evolution for the corresponding Kähler metric is given by:

$$\frac{\partial}{\partial t}\tilde{\omega}_t = -Ric(\tilde{\omega}_t) + Ric(\Omega) - e^{-t}\chi; \qquad \tilde{\omega}_t(0,\cdot) = \omega_0. \tag{1.2}$$

As pointed out by Zhang, the motivation is to apply the geometric flow techniques to study the complex Monge-Ampère equation:

$$(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n = \Omega. \tag{1.3}$$

This equation has already been intensively studied very recently by using the pluri-potential theory developed by Bedford-Taylor, Demailly, Kołodziej et al. When Ω is a smooth volume form and $[\omega_\infty]$ is Kähler, the equation is solved by Yau in his solution to the celebrated Calabi conjecture by using the continuity method [Y1]. When Ω is L^p with respect to another smooth reference volume form and $[\omega_\infty]$ is Kähler, the continuous solution is obtained by Kołodziej [K]. Later on, the bounded solution is obtained in [EGZ] and [Z2] independently, generalizing Kołodziej's theorem to the case when $[\omega_\infty]$ is big and semi-postive, and Ω is also L^p. On the other hand, as an interesting question, equation (1.3) is also studied on the symplectic manifolds by Weinkove [We].

In the case of the unnormalized Kähler-Ricci flow, the evolution for the cohomology class of the metrics is in the direction of the canonical class of the manifold. While in the case of (1.2), one can try to deform any initial metric class to an arbitrary desirable limit class. In particular, on Calabi-Yau manifolds, the flow (1.2) converges to a Ricci flat metric, if Ω is a Calabi-Yau volume form, with the initial metric also Ricci flat in a different cohomology class [Z3].

The existence and convergence of the solution are proved by Zhang [Z3] for the above flow in the case when $[\omega_\infty]$ is Kähler, which corresponds to the case considered by Cao in the classical Kähler-Ricci flow [C]. When $[\omega_\infty]$ is big, (1.2) may produce singularities at finite time $T < +\infty$. In this case, the local C^∞ convergence of the flow away from the stable base locus of $[\omega_T]$ was obtained under the further assumption that $[\omega_T]$ is semi-ample. When $[\omega_\infty]$ is semi-ample and big, he obtained the long time existence of the solution and important estimates and conjectured the convergence even in this more general setting. In this note, we give a proof to this conjecture. We will give some definitions before stating our theorem.

Definition 1.1 *$[\gamma] \in H^{1,1}(X, \mathbb{C})$ is semi-positive if there exists $\omega \in [\gamma]$ such that $\omega \geq 0$, and is big if $[\gamma]^n := \int_X \gamma^n > 0$.*

Definition 1.2 *A closed, positive $(1,1)$- current ω is called a singular Calabi-Yau metric on X if ω is a smooth Kähler metric away from an analytic subvariety $E \subset X$ and satisfies $Ric(\omega) = 0$ away from E.*

Definition 1.3 *A volume form Ω is called Calabi-Yau volume form if*

$$Ric(\Omega) = -\sqrt{-1}\partial\overline{\partial} \log \Omega = 0.$$

Theorem 1.1 *Let X be a Kähler manifold with a Kähler metric ω_0. Suppose that $[\omega_\infty] \in H^{1,1}(X, \mathbb{C}) \bigcap H^2(X, \mathbb{Z})$ is semi-positive and big. Then along the modified Kähler-Ricci flow (1.2), $\widetilde{\omega}_t$ converges weakly in the sense of current and converges locally in C^∞ norm away from the stable base locus of $[\omega_\infty]$ to the unique solution of the degenerate Monge-Ampère equation (1.3).*

Corollary 1.1 *When X is a Calabi-Yau manifold and Ω is a Calabi-Yau volume form, $\widetilde{\omega}_t$ converges to a singular Calabi-Yau metric.*

Remark 1.1 The singular Calabi-Yau metrics are already obtained in [EGZ] on normal Calabi-Yau Kähler spaces, and obtained by Song-Tian [ST2] and Tosatti [To] independently in the degenerate class on the algebraic Calabi-Yau manifolds. The uniqueness of the solution to the equation (1.3) in the degenerate case when $[\omega_\infty]$ is semi-positive and big has been studied in [EGZ] [Z2] [DZ] etc. In particular, a stability theorem is proved in [DZ] which immediately implies the uniqueness. In [To], Tosatti studied the deformation for a family of Ricci flat Kähler metrics, whose cohomology classes are approaching a big and nef class. So our deformations give different paths connecting non-singular and singular Calabi-Yau metrics.

Remark 1.2 As $[\omega_\infty] \in H^{1,1}(X, \mathbb{C}) \bigcap H^2(X, \mathbb{Z})$, there exists a line bundle L over X such that $\omega_\infty \in c_1(L)$. Moreover, L is big when $[\omega_\infty]$ is semi-positive and big [De]. Hence X is Moishezon. Furthermore, X is algebraic, for it is Kähler. Therefore, by applying Kodaira lemma, we see that for any small positive number $\epsilon \in \mathbb{Q}$, there exists an effective divisor E on X, such that $[L] - \epsilon[E] > 0$.

The structure of the paper is as follows: in the second section, we define an energy functional whose derivative in t along the flow is essentially bounded by the L^2 norm of the gradient of the Ricci potential, after deriving uniform estimates for the metric potential. From this property, we derive the convergence of the Ricci potential, and furthermore obtain the convergence of the flow. In the third section, we sketch a proof to the exponential convergence when $[\omega_\infty]$ is Kähler also by using the energy functional defined in the second section.

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2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1. We first define some notations for the convenience of our later discussions. Let $\widetilde{\Delta}$ and $\frac{\partial}{\partial t} - \widetilde{\Delta}$ be the Laplacian and the heat operator with respect to the metric $\widetilde{\omega}_t$ and let ∇ denote the gradient operator with respect to metric $\tilde{\omega}_t$. Let S be the defining section of E. By Kodaira Lemma as stated in Remark 1.2, there exists a hermitian metric h_E on E such that $\omega_\infty - \epsilon Ric(h_E)$, denoted by ω_E, is strictly positive for any ϵ small. Let $V_t = [\tilde{\omega}_t]^n$ with V_t uniformly bounded and $V_0 = 1$. Write $\dot{\varphi} = \frac{\partial \varphi}{\partial t}$ for simplicity.

Before proving the convergence, we would like to sketch the uniform estimates of φ.

First of all, by the standard computation as in [Z3], the uniform upper bound of $\frac{\partial}{\partial t}\varphi$ is deduced from the maximum principle. Secondly, by the result of [EGZ] [Z2], generalizing the theorem of Kołodziej [K] to the degenerate case, we have the C^0-estimate $\|u\|_{C^0(X)} \leq C$ independent of t where $u = \varphi - \int_X \varphi\Omega$. Then to estimate $\frac{\partial}{\partial t}\varphi$ locally, we will calculate $(\frac{\partial}{\partial t} - \widetilde{\Delta})[\dot{\varphi} + A(u - \epsilon \log \|S\|^2_{h_E})]$, and then the maximum principle yields $\frac{\partial}{\partial t}\varphi \geq -C + \alpha \log \|S\|^2_{h_E}$ for $C, \alpha > 0$.

Next, we follow the standard second order estimate as in [Y1] [C] [Si] [Ts].

Calculating $(\frac{\partial}{\partial t} - \widetilde{\Delta})[\log tr_{\omega_E + e^{-t}\chi}\widetilde{\omega}_t - A(u - \epsilon \log \|S\|^2_{h_E})]$ and applying maximum principle, we have: $|\ \Delta_{\omega_E}\varphi\ | \leq C$. Then by using the Schauder estimates and third order estimate as in [Z3], we can obtain the local uniform estimate: For any $k \geq 0, K \subset\subset X \setminus E$, there exists $C_{k,K} > 0$, such that:

$$\|u\|_{C^k([0,+\infty)\times K)} \leq C_{k,K}. \tag{2.1}$$

In [Z3], Zhang proved the following theorem by comparing (1.1) with the Kähler-Ricci flow (2.2). We include the detail of the proof here for the sake of completeness. This uniform lower bound appears to be crucial in the proof of convergence.

Theorem 2.1 *([Z3]) There exists $C > 0$ such that $\frac{\partial}{\partial t}\varphi \geq -C$ holds uniformly along (1.1) or (1.2).*

We derive a calculus lemma now for later application.

Lemma 2.1 *Let $f(t) \in C^1([0,+\infty))$ be a non-negative function. If $\int_0^{+\infty} f(t)dt < +\infty$ and $\frac{\partial f}{\partial t}$ is uniformly bounded, then $f(t) \to 0$ as $t \to +\infty$.*

Proof We prove this calculus lemma by contradiction. Suppose there exist a sequence $t_i \to +\infty$ and $\delta > 0$, such that $f(t_i) > \delta$. Since $\frac{\partial f}{\partial t}$ is uniformly bounded, there exist a sequence of connected, non-overlapping intervals I_i containing t_i with fixed length l, such that $f(t) \geq \frac{\delta}{2}$ over I_i. Then $\int_{\bigcup_i I_i} f(t)dt \geq \sum_i l\frac{\delta}{2} \to +\infty$, contradicting with $\int_0^{+\infty} f(t)dt < +\infty$. □

Let $\widehat{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}\phi$ and consider the Monge-Ampère flow, as well as its corresponding evolution of metrics:

$$\frac{\partial}{\partial t}\phi = \log\frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\phi)^n}{\Omega} - \phi; \qquad \phi(0,\cdot) = 0. \tag{2.2}$$

$$\frac{\partial}{\partial t}\widehat{\omega}_t = -Ric(\widehat{\omega}_t) + Ric(\Omega) - \widehat{\omega}_t + \omega_\infty; \qquad \widehat{\omega}_t(0,\cdot) = \omega_0. \tag{2.3}$$

The following theorem is proved in [Z1] and we include the proof here.

Theorem 2.2 *There exists $C > 0$, such that $\frac{\partial}{\partial t}\phi \geq -C$ uniformly along (2.2).*

Proof Standard computation shows that:

$$(\frac{\partial}{\partial t} - \widehat{\Delta})(\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial t^2}) \leq -(\frac{\partial\phi}{\partial t} + \frac{\partial^2\phi}{\partial t^2}).$$

Then the maximum principle yields:

$$\frac{\partial}{\partial t}(\phi + \frac{\partial\phi}{\partial t}) \leq Ce^{-t} \ \ and \ \ \frac{\partial}{\partial t}\phi \leq Ce^{-\frac{t}{2}},$$

which means that ϕ and $\phi + \frac{\partial\phi}{\partial t}$ are essentially decreasing along the flow, for example: $\frac{\partial}{\partial t}(\phi + Ce^{-t}) \leq 0$. As we have the similar estimate that ϕ is locally uniformly bounded away from E, then ϕ converges to some $\omega_\infty-$ plurisubharmonic function away from E, which also yields that the pointwise limit of $\frac{\partial\phi}{\partial t}$ is 0 away from E by Lemma 2.1. On the other hand, the weak convergence of (2.2) is obtained in [Ts] [TZha]. Suppose ϕ_∞ is the weak solution to the degenerate Monge-Ampère equation as the limit equation with $\phi + \dot{\phi} \to \phi_\infty$ away from E when $t \to +\infty$:

$$e^{\phi_\infty} = (\omega_\infty + \sqrt{-1}\partial\overline{\partial}\phi_\infty)^n \ \ \leftarrow \ \ e^{\phi+\dot{\phi}} = (\omega_t + \sqrt{-1}\partial\overline{\partial}\phi)^n.$$

Furthermore, from the pluri-potential theory [EGZ] [Z2] and equation (2.2), we know that $-C \leq \phi, \phi_\infty \leq C$ uniformly. With the essentially decreasing property, we know that away from E:

$$\phi + \dot{\phi} + Ce^{-t} \geq \phi_\infty \geq -C,$$

yielding $\dot{\phi} \geq -C$ away from E. The theorem is proved with $\dot{\phi}$ smooth on X.

□

We are now ready to give Zhang's proof to Theorem 2.1.

Proof of Theorem 2.1:
Fix $T_0 > 0$. Let $\kappa(t, \cdot) = (1 - e^{-T_0})\dot{\varphi} + u - \phi(t + T_0)$ with $\kappa(0, \cdot) \geq -C_1$. Then:

$$\begin{aligned}
\frac{\partial}{\partial t}\kappa(t, \cdot) &= \widetilde{\Delta}((1 - e^{-T_0})\dot{\varphi} + u) + \dot{u} - \dot{\phi}(t + T_0) - n + tr_{\widetilde{\omega}_t}\omega_{t+T_0} \\
&= \widetilde{\Delta}((1 - e^{-T_0})\dot{\varphi} + u - \phi(t + T_0)) + \dot{u} - \dot{\phi}(t + T_0) - n + tr_{\widetilde{\omega}_t}\widehat{\omega}_{t+T_0} \\
&\geq \widetilde{\Delta}((1 - e^{-T_0})\dot{\varphi} + u - \phi(t + T_0)) + \dot{\varphi} - C_2 + n(\frac{C_3}{e^{\dot{\varphi}}})^{\frac{1}{n}},
\end{aligned}$$

where the fact: $\dot{u} \geq \dot{\varphi} - C$, $-C \leq \frac{\partial}{\partial t}\phi \leq C$ and $\widehat{\omega}_t \geq C_3\Omega$ are used.

Suppose $\kappa(t, \cdot)$ achieves minimum at (t_0, p_0) with $t_0 > 0$. Then the maximum principle yields $\dot{\varphi}(t_0, p_0) \geq -C_4$. Hence $\dot{\varphi}$ is bounded from below. Suppose $\kappa(t, \cdot)$ achieves minimum at $t = 0$. Then the theorem follows trivially. □

Inspired from the Mabuchi K-energy in the study on the convergence of Kähler-Ricci flow on the Fano manifolds, we similarly define an energy functional as follows:

$$\nu(\varphi) = \int_X \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega}(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n.$$

Next, we will give some properties of $\nu(\varphi)$ and then a key lemma for the proof of the theorem.

Proposition 2.1 *$\nu(\varphi)$ is well-defined and there exists $C > 0$, such that $-C < \nu(\varphi) < C$ along (1.2).*

Proof It is easy to see that $\nu(\varphi)$ is well-defined. If we rewrite

$$\nu(\varphi) = \int_X \dot{\varphi}\widetilde{\omega}_t^n,$$

then $\nu(\varphi)$ is uniformly bounded from above and below by the upper and lower bound of $\dot{\varphi}$. Here, we derive the uniform lower bound by Jensen's inequality without using Theorem 2.1:

$$\nu(\varphi) = -V_t \int_X \log \frac{\Omega}{\widetilde{\omega}_t^n} \frac{\widetilde{\omega}_t^n}{V_t} \geq -V_t \log \int_X \frac{\Omega}{V_t} \geq -C,$$

as V_t is uniformly bounded. □

Proposition 2.2 *There exists constant $C > 0$ such that for all $t > 0$ along the flow (1.2):*

$$\frac{\partial}{\partial t}\nu(\varphi) \leq -\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n + Ce^{-t}. \tag{2.4}$$

Proof Along the flow (1.2), we have:

$$\begin{aligned}
\frac{\partial}{\partial t}\nu(\varphi) &= \int_X \tilde{\Delta}\dot{\varphi}\widetilde{\omega}_t^n - e^{-t}\int_X tr_{\widetilde{\omega}_t}\chi\widetilde{\omega}_t^n - ne^{-t}\int_X \dot{\varphi}\chi\wedge\widetilde{\omega}_t^{n-1} + n\int_X \dot{\varphi}\sqrt{-1}\partial\overline{\partial}\dot{\varphi}\wedge\widetilde{\omega}_t^n \\
&= -\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n - ne^{-t}\int_X \chi\wedge\widetilde{\omega}_t^{n-1} - ne^{-t}\int_X \dot{\varphi}\chi\wedge\widetilde{\omega}_t^{n-1} \\
&= -\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n - n[\chi][\omega_t]^{n-1}e^{-t} - ne^{-t}\int_X \dot{\varphi}(\omega_0-\omega_\infty)\wedge\widetilde{\omega}_t^{n-1} \\
&\leq -\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n + n[\chi][\omega_t]^{n-1}e^{-t} + Ce^{-t}\int_X (\omega_0+\omega_\infty)\wedge\omega_t^{n-1} \\
&= -\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n + [n\chi + C\omega_0 + C\omega_\infty][\omega_t]^{n-1}e^{-t} \\
&\leq -\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n + C'e^{-t}.
\end{aligned}$$

Notice that we used the evolution of $\dot{\varphi}$ and integration by parts in the first two equalities and the uniform bound of $\dot{\varphi}$ in the first inequality and the last inequality holds since ω_t is uniformly bounded. □

Lemma 2.2 *On each $K \subset\subset X \setminus E$, $\|\nabla\dot{\varphi}(t)\|^2_{\widetilde{\omega}_t} \to 0$ as $t \to +\infty$ uniformly.*

Proof Integrating (2.4) from 0 to T, we have:

$$-C \leq \nu(\varphi)(T) - \nu(\varphi)(0) \leq -\int_0^T\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n dt + C$$

for some constant $C > 0$. It follows that

$$\int_0^{+\infty}\int_X \|\nabla\dot{\varphi}\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n dt \leq 2C,$$

by letting $T \to +\infty$. Hence, (2.1) and Lemma 2.1 imply that for any compact set $K' \subset X \setminus E$,

$$\int_{K'} \|\nabla\dot{\varphi}(t,\cdot)\|^2_{\widetilde{\omega}_t}\widetilde{\omega}_t^n \to 0. \tag{2.5}$$

Now, assume that there exists $\delta > 0$, $z_j \in K$ and $t_j \to \infty$ such that $\|\nabla\dot{\varphi}(t_j, z_j)\|^2_{\widetilde{\omega}_{t_j}} > \delta$. It follows from (2.1) that $\|\nabla\dot{\varphi}(t_j, z)\|^2_{\widetilde{\omega}_{t_j}} > \frac{\delta}{2}$ for $z \in B(z_j, r) \subset K'$ for $K \subset K' \subset\subset X \setminus E$ and some $r > 0$. This contradicts with (2.5).

□

Proof of Theorem 1.1:

First of all, we want to show that for any $k \in \mathbb{Z}$, $K \subset\subset X \setminus E$, $u(t) \to \psi$ in $C^\infty(K)$.

Exhaust $X\setminus E$ by compact sets K_i with $K_i \subset K_{i+1}$ and $\bigcup_i K_i = X\setminus E$. As $\|u\|_{C^k(K_i)} \leq C_{k,i}$, after passing to a subsequence t_{ij}, we know $u(t_{ij}) \to \psi$ in $C^\infty(K_i)$ topology. By picking up the diagonal subsequence of $u(t_{ij})$, we know that $\psi \in L^\infty(X)\bigcap C^\infty(X \setminus E)\bigcap PSH(X \setminus E, \omega_\infty)$. Furthermore, ψ can be extended over E as a function in $PSH(X, \omega_\infty)$. Taking gradient of (1.1), by Lemma 2.2, we have on K_i as $t_{ij} \to +\infty$,

$$\nabla\dot{\varphi} = \nabla \log \frac{(\omega_t + \sqrt{-1}\partial\overline{\partial}\varphi)^n}{\Omega} \to 0 = \nabla \log \frac{(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\Omega}.$$

Hence, we know that $\log \frac{(\omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi)^n}{\Omega} = constant$ on $X \setminus E$. Then the constant can only be 0 as ψ is a bounded pluri-subharmonic function and $\int_X \omega_\infty^n = \int_X \Omega$, which means that ψ solves the degenerate Monge-Amperè equation (1.3) globally in the sense of current and strongly on $X \setminus E$. Furthermore, we notice that $\int_X \psi\Omega = 0$ as ψ is bounded.

Suppose $u(t) \nrightarrow \psi$ in $C^\infty(K)$ for some compact set $K \subset X \setminus E$, which means that there exist $\delta > 0$, $l \geq 0$, $K' \subset\subset X \setminus E$, and a subsequence $u(s_j)$ such that $\|u(s_j) - \psi\|_{C^l(K')} > \delta$. While $u(s_j)$ are bounded in $C^k(K)$ for any compact set K and $k \geq 0$, from the above argument, we know that by passing to a subsequence, $u(s_j)$ converges to ψ' in $C^\infty(K)$ for any $K \subset\subset X \setminus E$, where ψ' is also a solution to equation (1.3) under the normalization $\int_X \psi'\Omega = 0$, which has to be ψ by the uniqueness of the solution to (1.3). This is a contradiction. It thus follows that $u(t) \to \psi$ in $L^p(X)$ for any $p > 0$.

Notice that $u(t), \psi$ are uniformly bounded. Integrating by part, we easily deduce that $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}u \to \omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi$ weakly in the sense of current. The proof of the theorem is complete.

$\square$

3 Remarks on non-degenerate case

In the case when $[\omega_\infty]$ is Kähler, the convergence of (1.2) has already been proven by Zhang in [Z3] by modifying Cao's argument in [C]. However, by using the functional $\nu(\varphi)$ defined in the previous section, we will have an alternating proof to the convergence, without using Li-Yau's Harnack inequality. We will sketch the proof in this section. We believe that this point of view is well-known to the experts.

Firstly, under the same normalization $u = \varphi - \int_X \varphi\Omega$, we will have the following uniform estimates ([Z3]): for any integer $k \geq 0$, there exists $C_k > 0$, such that

$$\|u\|_{C^k([0,+\infty)\times X)} \leq C_k.$$

Secondly, following the convergence argument as in the previous section, we will obtain the C^∞ convergence of $\tilde{\omega}_t$ along (1.1). More precisely, $u \to \psi$ in C^∞-norm with ψ solving (1.3) as a strong solution. In particular, $\dot{\varphi} \to 0$ in C^∞-norm as $t \to +\infty$. Let $\tilde{\omega}_\infty = \omega_\infty + \sqrt{-1}\partial\overline{\partial}\psi$ be the limit metric. Furthermore, we have the bounded geometry along (1.2) for $0 \leq t \leq +\infty$:

$$\frac{1}{C}\tilde{\omega}_\infty \leq \tilde{\omega}_t \leq C\tilde{\omega}_\infty. \tag{3.1}$$

Finally, we need to prove the exponential convergence: $\|\tilde{\omega}_t - \tilde{\omega}_\infty\|_{C^k(X)} \leq C_k e^{-\alpha t}$ and $\|u(t) - \psi\|_{C^k(X)} \leq C_k e^{-\alpha t}$, for some $C_k, \alpha > 0$. Then it is sufficient to prove: for any integer $k \geq 0$, there exists $c_k > 0$, such that

$$\|D^k\dot{\varphi}\|^2_{\omega_0} \leq c_k e^{-\alpha t}.$$

Essentially, by following the proof of Proposition 10.2 in the case of holomorphic vector fields $\eta(X) = 0$ in [CT], we can also prove the following proposition.

Proposition 3.1 *Let* $c(t) = \int_X \frac{\partial\varphi}{\partial t}\tilde{\omega}_t^n$. *There exists* $\alpha > 0$ *and* $c'_k > 0$ *for any integer* $k \geq 0$, *such that*

$$\int_X \|D^k(\frac{\partial\varphi}{\partial t} - c(t))\|^2_{\tilde{\omega}_t}\tilde{\omega}_t^n \leq c'_k e^{-\alpha t}.$$

By following the argument of Corollary 10.3 in [CT], we can prove that $|c(t)| \leq Ce^{-\alpha t}$. On the other hand, since the geometry is bounded along the flow (3.1), the Sobolev constants are uniformly bounded. By using the Sobolev inequality, we have:

$$\|D^k(\dot{\varphi} - c(t))\|^2_{\omega_0} \leq c_k e^{-\alpha t}.$$

The exponential convergence is thus obtained combining the estimate of $c(t)$.

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